Adaptive Predictive Control of Fractional Order Chaotic Systems

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ABSTRACT

In this paper, an adaptive predictive control for controlling and stabilizing fractional order chaotic systems around the equilibrium point is provided. The stability of fractional order chaotic systems around equilibrium points in the presence of parameter uncertainty has been demonstrated using Lyapunov’s stability theorem. In addition, the uncertain parameters of fractional order chaotic systems are calculated using appropriate adaptive methods based on the proposed predictive controller structure. Rossler and Chen systems were considered to numerical simulations. The results demonstrated the adaptive predictive control method’s usefulness and performance.

Index Terms— Adaptive control, equilibrium point, fractional order chaotic systems, predictive control.

I. INTRODUCTION

Chaotic systems are nonlinear systems that are extremely sensitive to initial conditions and are difficult to predict. These systems have been observed in a variety of applications including chemistry, biology, finance, and engineering. Many control strategies, such as adaptive control [1–5], sliding mode control [6, 7], adaptive back-stepping control [8], and predictive and fuzzy control [9], have been applied in research to govern chaotic systems. In recent decades, scholars have been particularly interested in fractional calculations. These calculations are related to non-integer derivatives (fractional). Because fractional calculations have a memory and inheritance property, various systems are explained more correctly than integer order models. Controlling chaotic systems with a fractional order model is one of the newest topics of interest among researchers. Many controllers were utilized in the research to control fractional chaotic systems.

The followings are among the studies on the use of predictive control to govern the behavior of fractional chaotic systems. In [10], the chaotic system is controlled by a fuzzy and predictive controller. The authors of [11] examined the dynamics of a fractional order chaotic system, its stabilization via predictive control, and its circuit validation. In 2019, Zoad et al. [12] used predictive control of a fractional order delayed chaotic system with its circuit implementation.

Wang et al. [13], in 2018, proposed a nonlinear fuzzy predictive control for a class of integer order chaotic systems. In 2017, Khan et al. [14] employed a predictive controller to control the Rabinovich chaotic system. The control input was arranged in such a way that the chaotic trajectories converged on the unstable equilibrium points. A practical predictive control model for a set of noisy chaotic hybrid systems connected to the Chua circuit was proposed in [15]. Wang et al. suggested a fuzzy generalized predictive control for fractional order nonlinear systems in 2017 [16]. Based on the Grünwald–Letnikov definition, Laplace transform, and discretization, a group of fractional order nonlinear systems was translated to the autoregressive community moving average (CARMA) model in the stated article. A linear CARMA model for nonlinear systems was presented based on Takagi–Sugeno’s fuzzy theory. Then, using the CARMA predictive model and generalized predictive control theory, a generalized predictive control approach for fractional order nonlinear systems was proposed. To control chaotic systems, [17] employed a robust fractional order controller based on predictive control. The simulation results on three-dimensional Lorenz and Chen chaotic systems demonstrated the efficacy of the robust control technique. In [18], Zheng and Li used predictive control to govern fractional order systems.
A. Predictive Feedback Controller Design

The controlled system is defined as follows:

\[ D^\alpha x(t) = f(x(t)) + u(t) \]  

(5)

In this article, the parameter estimation rules are constructed using Lyapunov's stability theorem, and the system states converge to the equilibrium point using predictive control.

The following is the rest of this article. The basic concepts and preliminary steps for fractional calculations are covered in the second section. The third section describes how to develop predictive and adaptive control for fractional order chaotic systems. The fourth section includes numerical simulations. The fifth section contains conclusions and recommendations for future research in this topic.

II. PRELIMINARIES

For fractional calculus, there are three well-known definitions: Caputo, Riemann–Liouville, and Grünwald–Letnikov. For fractional calculations, the Grünwald–Letnikov definition is used in this article. The fractional derivative using Grünwald–Letnikov concept is as follows [28]:

\[ \mathcal{D}^\alpha_{t0} f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{t-t_0}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh), \]  

(1)

where \( \mathcal{D}^\alpha_{t0} f(t) \) is the derivative-integral operator of fractional order with the order \( \alpha (\alpha \in \mathbb{R}) \), which means the integer part, \( (t - a) / h \), and \( a \) and \( t \) are operational ranges for the operator \( \mathcal{D}^\alpha_{t0} f(t) \). The expression

\[ \binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \]  

(2)

where \( \Gamma \) is the gamma function.

For numerical simulation, the modified version of Eq. (2) is used. The numerical approximation of \( \alpha \) at \( kh \) points (\( k=1, 2, \ldots \)) is as follows.

\[ (k-L_m / h)\mathcal{D}^\alpha_{t0} f(t) = \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(t_j - j), \]  

(3)

where \( L_m \) is the memory length, \( t_j = kh \), \( h \) is the time step, \( (-1)^j \binom{\alpha}{j} \) are binomial coefficients, and \( c_i^0 (j = 0, 1, 2, \ldots) \) are

\[ c_i^0 = 1, c_i^0 = (1 - \frac{1+\alpha}{j}) c_i^{\alpha}, \]  

(4)

III. PROBLEM STATEMENT AND CONTROLLER DESIGN

In this part, first, a predictive feedback control is devised to stabilize a fractional order chaotic system at the equilibrium point by introducing a control signal to the fractional order chaotic system. The adaptive controller is then employed to estimate the uncertain parameters of the fractional order chaotic system. In this paper, it has been assumed states of the system are observable.

A. Predictive Feedback Controller Design

The controlled system is defined as follows:

\[ D^\alpha x(t) = f(x(t)) + u(t) \]  

(5)

Because the parameters of chaotic systems are often unknown in practice, an adaptive control technique is required to estimate the system parameters. The predictive controller was also adopted because of its ability to predict the system's future behavior. As a result, the fundamental contribution of this paper is the combination of predictive control and adaptive control to control the behavior of chaotic systems with a fractional order model toward the equilibrium point. A previous study indicates that the combination of these two controllers has not been examined in this application.
Assume that the control law is as follows:

$$u(t) = k \left( D^x f(x(t)) + x(t) - x_r \right) = k \left( f(x(t)) + x(t) - x_r \right),$$  \hspace{1cm} (6)

in which $x_r$ is an unstable equilibrium point of the system (5) and $k$ is a negative control parameter such that $k \neq -1$. By substituting Eq. (6) in Eq. (5), we have:

$$D^x x(t) = \dot{f}(x(t)) = (1 + k)f(x(t)) + k(x(t) - x_r)$$  \hspace{1cm} (7)

Assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the Jacobian matrix $(D)$ of the system (5) in the equilibrium point $x_r$ without controller. Also assume that $\sigma_j\beta_j$ are the real and imaginary parts of the eigenvalues of the Jacobian matrix of system (5) without controller. The authors of [18] have shown that the following equation holds:

$$\bar{k} = \min \left\{ \frac{-\sigma_j \tan \frac{\alpha \pi}{2} + |\beta_j|}{(1 + \sigma_j) \tan \frac{\alpha \pi}{2} - |\beta_j|} \right\} \arg(\sigma_j + \beta_j) \leq \frac{\alpha \pi}{2}. \hspace{1cm} (8)$$

Assuming that $\bar{k}$ is equal to Eq. (8), it was shown in [18] that if $k = (-1, \bar{k})$, then the equilibrium point, $x_r \in R^n$, of the controlled system (5) is asymptotically locally stable. To illustrate this, consider the following theorem:

**Theorem 1** [18]: If the control law $u(t)$ is considered as Eq. (6), then the system of Eq. (7) will be stable towards the equilibrium point, $x_r$, considering $k \neq -1$.

Note 1: Since the stability around the equilibrium point is guaranteed, the predictive control law is defined as the following switching law:

$$u(t) = \begin{cases} k(f(x(t)) + x(t) - x_r) & \text{if } \|x(t-1) - x(t)\| < \varepsilon \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (9)

where $\varepsilon$ is a small positive number such that it meets conditions of Eq. (9).

A Lyapunov function and its derivative are considered as follows:

$$V = \frac{1}{2} e^T D^x V = e D^x e$$  \hspace{1cm} (10)

In which, the control error toward the equilibrium point, $e = x - x_r$, is zero. By applying of the controller (6) to system (5) and applying the fractional order derivative to the Lyapunov function in Eq. (10), the following equation is obtained.

$$D^x V = e(f(x(t) + k) e)$$  \hspace{1cm} (11)

where $\eta$ is a small positive number. The Lyapunov's derivative is as follows:

$$D^x V = e(f(x(t)) + k) e \leq e(ke + \eta f(x)) < 0,$$  \hspace{1cm} (12)

where $\eta$, $(x)$ is a small number close to zero. Therefore, in terms of the Lyapunov stability criterion, the stability conditions are also established because of $\eta$ being small. Now, if there are parameters with uncertainty in the system (5), this control method must have fundamental changes, therefore, the innovation of this article is the use of the adaptive control plan in conditions where the system parameters are unknown.

Remark. Control law in Eq. (6) can be applied in practical applications. The controller is generally a negative feedback of the observed system states $x(t)$ and the known equilibrium point $x_r$, is a negative expression and it is calculated by implementation of predictive control. The expression $f(x(t))$ may have unknown parameters that will be obtained by using adaptive rules.

B. Designing of the Adaptive Controller Combined With Predictive Control

According to the predictive controller designed in the previous section, an adaptive controller is proposed in this section to estimate the unknown parameters of fractional order chaotic systems.

It is considered that a hyper-chaotic system with a controller can be generalized using Eq. (13).

$$\begin{align*}
D^x x_1 &= A_1 x + g_1(x) + u_1(t) \\
D^x x_2 &= A_2 x + g_2(x) + u_2(t) \\
D^x x_3 &= A_3 x + g_3(x) + u_3(t) \\
D^x x_4 &= A_4 x + g_4(x) + u_4(t)
\end{align*}$$  \hspace{1cm} (13)

In which $A_i, A_2, A_3, A_4$ include uncertain parameters. The terms $g_i(x), i = 1, 2, 3, 4$, and $u_i(t)$ include the state variables without parametric uncertainty and control efforts, respectively. The control efforts are calculated as in Eq. (14).

$$\begin{align*}
u_1(t) &= k_1 \dot{A}_1 x + g_1(x) + (x_1 - x_{r_1}) \\
u_2(t) &= k_2 \dot{A}_2 x + g_2(x) + (x_2 - x_{r_2}) \\
u_3(t) &= k_3 \dot{A}_3 x + g_3(x) + (x_3 - x_{r_3}) \\
u_4(t) &= k_4 \dot{A}_4 x + g_4(x) + (x_4 - x_{r_4})
\end{align*}$$  \hspace{1cm} (14)

In Eq. (14), $k_i, j = 1, 2, 3, 4$ are negative. $\dot{A}_1, \dot{A}_2, \dot{A}_3, \dot{A}_4$ are parameter estimations of system (13) determined in Eq. (18). The estimation error is defined as $\dot{A}_i - A_i, i = 1, 2, 3, 4$. The $x_i$ parameters are the components of the equilibrium point and equal to zero.

**Theorem 2**: If the control signal (14) is applied to the system (13) along with the parameter estimation criteria in Eq. (18), the system (13) will tend to the equilibrium point asymptotically, according to the Lyapunov stability theorem.

Proof: The Lyapunov function is used to control the system toward the equilibrium point as Eq. (15).

$$V = \frac{1}{2} e_1^2 + \frac{1}{2} e_2^2 + \frac{1}{2} e_3^2 + \frac{1}{2} e_4^2 + \frac{1}{2} \dot{A}_1^2 + \frac{1}{2} \dot{A}_2^2 + \frac{1}{2} \dot{A}_3^2 + \frac{1}{2} \dot{A}_4^2$$  \hspace{1cm} (15)

The proposed Lyapunov function’s derivative is as in Eq. (16):
\[ D^n V = e_i(D^n e_i) + e_j(D^n e_j) + e_k(D^n e_k) + e_l(D^n e_l) + \lambda_i e_i + \lambda_j e_j + \lambda_k e_k + \lambda_l e_l, \]

where \( \lambda_i < 0, i = 1, 2, 3, 4 \). By applying the parameter estimation laws and considering that \( x(1 + k_i)A_i \), \( i = 1, 2, 3, 4 \) is close to zero, the derivative of the Lyapunov function becomes as follows:

\[ D^n V = k_i e_i + \lambda_i e_i, \quad i = 1, 2, 3, 4 \]

As a result, the fractional derivative of Lyapunov function is negative, and the parameters are determined using Eq. (18). Fig. 1 depicts the block diagram of the adaptive predictive control method for chaotic system control.

As shown in Fig. 1, the uncertain parameters are estimated in the adaptive design block before entering the predictive control block and calculating the control effort vectors. The control effort is then applied to the chaotic system in order to stabilize and converge the system states toward the equilibrium point. The states of \( X \) are calculated and compared to the reference value to determine the error. The pre-control procedure between the systems then continues using the obtained error and the estimated parameters. This process is repeated until the simulation time is over and the states converge to the equilibrium point.

In the following, the simulations and the results on two fractional order chaotic systems of Rossler and the Chen hyper-chaotic system.

IV. SIMULATIONS AND RESULTS

The proposed adaptive predictive controller was applied to Rossler chaotic system and Chen hyperchaotic systems in this section, and the results were obtained. The simulation findings for Rossler system are presented first, followed by the simulation results for Chen’s hyperchaotic system.

A. Rossler System

The fractional order Rossler system is described as follows:

\[
\begin{align*}
D^n x_1(t) &= -x_2(t) - x_3(t), \\
D^n x_2(t) &= x_1(t) + ax_2(t), \\
D^n x_3(t) &= bx_3(t) - (c - x_1(t))x_3(t),
\end{align*}
\]

This system with order \( \alpha = 0.97 \) and \( a = 0.34, b = 0.4, c = 4.5 \) shows chaotic behavior [18].

Consider the system with controller \( u(t) = [u_1 \ u_2 \ u_3]^T \), as follows:

\[
\begin{align*}
D^n x_1(t) &= -x_2(t) - x_3(t) + u_1(t), \\
D^n x_2(t) &= x_1(t) + ax_2(t) + u_2(t), \\
D^n x_3(t) &= bx_3(t) - (c - x_1(t))x_3(t) + u_3(t),
\end{align*}
\]

Assume that the parameters \( a, b \), and \( c \) are uncertain and must be estimated. Control errors relative to the equilibrium point are defined as \( e_i = x_i - x_i, i = 1, 2, 3 \). It is shown in [18] that \( x_i \), is at the origin.

In this system it is defined that \( \tilde{a} = \hat{a} - a, \tilde{b} = \hat{b} - b, \tilde{c} = \hat{c} - c \). Therefore, \( D^n \tilde{a} = D^n \hat{a} - D^n a, D^n \tilde{b} = D^n \hat{b} - D^n b, D^n \tilde{c} = D^n \hat{c} - D^n c \). According to Theorem 2, the controllers can be calculated as follows:

\[
\begin{align*}
u_1 &= k_1(-x_2 - x_3 + e_1) \\
u_2 &= k_2(x_1 + \hat{a}x_2 + e_2) \\
u_3 &= k_3(\hat{b}x_3 - \hat{c}x_3 + x_3 + e_3)
\end{align*}
\]

where \( k_1, k_2, k_3 = -0.95 \). Estimation rules for unknown parameters are written as follows:
\[ D^\alpha \ddot{a} = -k_1 \dot{x}_2(x_2) + \lambda_a e_a \]
\[ D^\alpha \ddot{b} = -k_2 \dot{x}_3 + \lambda_b e_b \]
\[ D^\alpha \ddot{c} = k_3 \dot{x}_3 + \lambda_c e_c \]
where \( \lambda_a, \lambda_b, \) and \( \lambda_c = -0.95. \)

**B. Fractional Order Hyperchaotic Chen System**

Consider the fractional order hyperchaotic Chen system, which is as follows:

\[
\begin{align*}
\frac{d}{dt} x_1(t) &= ax_2(t) - x_1(t) + x_4(t), \\
\frac{d}{dt} x_2(t) &= dx_1(t) - x_1(t)x_2(t) + cx_2(t), \\
\frac{d}{dt} x_3(t) &= x_1(t)x_2(t) - bx_3(t), \\
\frac{d}{dt} x_4(t) &= x_1(t)x_3(t) + rx_4(t),
\end{align*}
\]

The system with control input is defined as follows:

\[
\begin{align*}
\frac{d}{dt} x_1(t) &= ax_2(t) - x_1(t) + x_4(t) + u_1(t), \\
\frac{d}{dt} x_2(t) &= dx_1(t) - x_1(t)x_2(t) + cx_2(t) + u_2(t), \\
\frac{d}{dt} x_3(t) &= x_1(t)x_2(t) - bx_3(t) + u_3(t), \\
\frac{d}{dt} x_4(t) &= x_1(t)x_3(t) + rx_4(t) + u_4(t),
\end{align*}
\]

Chaos control errors are \( e_i = x_i, \ i = 1, 2, 3, 4. \) Uncertain parameters include \( a, b, c, d, \) and \( r \) and must be estimated. In this system, parameter estimation errors are as \( \dot{\alpha} = \dot{\alpha} - a, \dot{\beta} = \dot{\beta} - b, \dot{\gamma} = \dot{\gamma} - c, \dot{\delta} = \dot{\delta} - d, \dot{\tau} = \dot{\tau} - r. \)

According to Theorem 2, the control efforts are calculated as follows:

\[
\begin{align*}
u_1 &= k_1(e_1 + \dot{\alpha}(x_2 - x_3) + x_4) \\
u_2 &= k_2(e_2 + \dot{\beta}x_1(x_1 + x_3) + \dot{\gamma}x_2) \\
u_3 &= k_3(e_3 + \dot{\delta}x_2(x_1 - \dot{\beta}x_1)) \\
u_4 &= k_4(e_4 + \dot{\tau}x_3(x_1 - \dot{\beta}x_1))
\end{align*}
\]

In which \( k_1 = k_2 = k_3 = k_4 = -0.95 \) are the predictive control gains. According to Eq. (18), the adaptive laws of the parameters are determined as follows:
The simulation results of the predictive control combined with adaptive control for fractional order Rossler system and fractional order hyperchaotic Chen system are given.

### C. Simulation Results of the Fractional Order Rossler System

The simulation results for the fractional order chaotic Rossler system are shown in this section. The simulation time is 100 seconds, and the controller is activated after 50 seconds. The time step size is 0.01 and the fractional order parameter is assumed to be 0.97. Fig. 2 depicts the results of the control of state signals $x_1, x_2,$ and $x_3$. The parameter $k$ is assumed to be $-0.95$.

It can be seen in Fig. 1 that after 50 seconds, $x_1(t)$ quickly reaches zero by applying the controller. Fig. 2 shows the estimation results of $\hat{a}, \hat{b}, \hat{c}$.

Fig. 3 shows that the parameters are well estimated from the start using Eq. (23) and without the control signal. And they changed from their original value at the 50s when the controller was applied, but they returned to their original value after the controller was removed.

### D. Simulation Results of the Fractional Order Hyperchaotic Chen System

In this section, the results related to predictive control combined with adaptive control for Chen fractional order hyperchaotic system are shown. In this case, the simulation time is equal to 100 seconds,
the time step size is 0.01, and the controller is applied at 50s. The results of applying the proposed controller are shown in Fig. 4 related to the states $x_1, x_2, x_3$. The controller gains $k_1, k_2, k_3$ were considered equal to -0.95. Control gains of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are -0.95 are considered.

It can be seen from Fig. 4 that since the controller is applied, the system states have converged to the zero-equilibrium point. Fig. 5 and Fig. 6 show the parameter estimation results.

Figs. 5 and 6 show that the parameters are also estimated once the controller is applied. According to the results, the proposed design is capable of controlling fractional order Chen and Rossler systems and estimating their parameters.

**V. CONCLUSION**

The control problem of Rossler and Chen chaotic systems was investigated in this paper using the predictive feedback approach along with the estimation of unknown system parameters using adaptive control. The control of chaotic and hyperchaotic systems toward equilibrium points was researched using Lyapunov’s stability theory. The adaptive rules and controllers were then obtained with its assistance, allowing a set of fractional order chaotic systems to converge toward their equilibrium points. According to the results, all of the modes have converged toward the equilibrium point, and the parameters have also been appropriately determined. The practical application of the suggested adaptive predictive controller for synchronizing these systems while accounting for uncertainty and unknown parameters can be explored for future research in this subject. The practical application of the proposed controller can be used for Chua circuit. It is also proposed that the mentioned controller be used to control and synchronize chaotic and hyperchaotic systems with delayed fractional order.

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